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Probabilistic solutions of generalised birth and death equations and applications to field theory

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Abstract. In this paper we give probabilistic solutions to the equations describing non-relativistic quantum 'electrodynamical' systems.

These solutions involve, besides the usual diffusion processes, also birth and death processes corresponding to the 'photon number' variables. We state some inequalities and in particular we establish bounds to the ground-state energy of systems composed of a non-relativistic particle interacting with a field. The result is general and it is applied as an example to the polaron problem.

1. Introduction

In the last few years the overlap between quantum mechanics and the theory of stochastic processes has increased enormously. On the one side there have been successful attempts to give a stochastic description of some quantum phenomena and, on the other side, new probabilistic methods have been introduced both in the functional integral approach to quantum mechanics and in the quantisation procedure known as 'stochastic quantisation'. These different applications of stochastic processes are sometimes related.

Consider for example Nelson's stochastic mechanics; it is a theory that gives a naive microphysical picture of systems described by the Schrödinger equation, but it turns out that the mathematical apparatus can be regarded as a probabilistic version of the Feynman path integral (Guerra 1981), in other words the ground-state process of Nelson's stochastic mechanics solves the imaginary time Schrödinger equation.

More recent papers have shown, on the one hand, that it is possible, utilising discrete processes, to give a stochastic description of spinning particles (De Angelis and Jona-Lasinio 1982) and of relativistic particles (De Angelis *et al* 1984) and, on the other hand, that Feynman-Kac formula could be extended to Pauli (De Angelis *et al* 1983) and to Dirac equations (Blanchard *et al* 1984)

Even in this case it has been pointed out that the processes associated with stochastic models were closely related to the processes utilised to obtain the generalised Feynman-Kac formulae.

It is natural at this point to follow this line and to try to investigate other possible relations between stochastic mechanics models and the functional integral approach to quantum mechanics.

Here we explore the possibility of giving probabilistic solutions to field equations in the occupation number representation. The starting point is a recent paper (Cini and Serva 1984) where we gave a stochastic description of a simple field model based on the use of a birth and death type variable representing the number of photons. We use the process associated to the ground state of this physical model to obtain an analogue of the Feynman–Kac formula, where the paths of Wiener variables associated to the positions of the field's oscillators are replaced by paths of discrete variables representing the number of 'photons'.

The paper is organised as follows. In § 2 we introduce the process treating the simple case of a time-independent forced harmonic oscillator. We consider the associated Schrödinger equation in the occupation number representation and, after a canonical transformation of the wavefunction, we obtain the solution as an expectation with respect to a birth and death process. In § 3 we use this result to treat the more complicated case of a field in interaction with a particle. Here we take the expectation both with respect to birth and death variables associated to field oscillators, and with respect to a Wiener variable associated to the particle position. We then state an inequality that permits us to give a lower bound for the ground-state energy of the system. In § 4 we eliminate the particle variables in order to be left only with oscillator variables. Moreover we state other inequalities and we find an upper bound for the ground-state energy. The bounds cover a large number of different physical field–particle interactions; as an example they are tested for the well known polaron interaction.

2. The forced harmonic oscillator

The method we want to introduce finds its natural application in field theory. It is therefore instructive to start with the simple case of the one-dimensional forced harmonic oscillator. The Hamiltonian can be written

$$H=\tfrac{1}{2}(p^2+\omega^2q^2)+\omega(2\omega)^{1/2}\lambda q \tag{2.1}$$

where q and p are the position and the momentum operators of the oscillators and λ is a real constant.

Having defined

$$a^+=\frac{1}{(2\omega)^{1/2}}(\omega q-\mathrm{i}p) \qquad a=\frac{1}{(2\omega)^{1/2}}(\omega q+\mathrm{i}p) \tag{2.2}$$

it is easy to realise that the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_t\rangle=H|\psi_t\rangle \tag{2.3}$$

can be rewritten in the form

$$\mathrm{i}\frac{\partial}{\partial t}\psi(n,t)=(n+\tfrac{1}{2})\omega\psi(n,t)+\lambda\omega(n+1)^{1/2}\psi(n+1,t)+\lambda\omega(n)^{1/2}\psi(n-1,t) \tag{2.4}$$

where $\psi(n,t)=\langle n|\psi_t\rangle$ is the probability amplitude of finding the system in a state with n 'photons'.

Let us perform the canonical substitution $\psi(n,t)\to\phi(n,t)=\psi(n,t)/\Omega(n,t)$ where $\Omega(n,t)$ is the ground-state solution given by

$$\Omega(n,t)=\frac{(-\lambda)^n}{(n!)^{1/2}}\exp(-\tfrac{1}{2}\lambda^2-\mathrm{i}E_0t) \tag{2.5}$$

with $E_0 = -\lambda^2\omega + \frac{1}{2}\omega$. For the new function $\phi(n, t)$ equation (2.4) can be rewritten as

$$i\frac{\partial}{\partial t}\phi(n, t) = (n+\lambda^2)\omega\phi(n, t) - \lambda^2\omega\phi(n+1, t) - n\omega\phi(n-1, t). \qquad (2.6)$$

The above relation has (after an analytical continuation in the time variable) the structure of a backward Kolmogorov equation for a discontinuous Markow process in the state space of positive integers. The intuitive interpretation of the process is clear; we have a 'source' that emits 'photons' with intensity $\lambda^2\omega$ independent of the number already emitted or absorbed and absorbs with intensity ωn proportional to the number of 'photons' present. The transition probability $p(m, t; n, t')$ that the system goes from n to m in the time $t - t'$ has been found in a previous paper (Cini and Serva 1984) and is given by

$$p(m, t; n, t') = \exp\{\lambda^2(\exp[-\omega(t-t')]-1)\}\Bigg(\sum_{k\leq n,m}\binom{m}{k}\binom{n}{k}\frac{k!}{m!}\exp[-k\omega(t-t')]$$
$$\times\{1-\exp[\omega(t-t')]\}^{m-k}\{1-\exp[-\omega(t-t')]\}^{n-k}(\lambda^2)^{m-k}\Bigg). \qquad (2.7)$$

We are thus able to write the solution of (2.6) for imaginary time

$$\phi(n, t) = \sum_{m=0}^{\infty} p(m, t; n, 0)\phi_0(m) \qquad (2.8)$$

where $\phi_0(n) = \phi(n, 0)$.

Equation (2.8) can be rewritten as an expectation with respect to the birth and death process $N_n(t)$ associated to the probability (2.7) and satisfying $N_n(0) = n$:

$$\phi(n, t) = \mathbb{E}[\phi_0(N_n(t))]. \qquad (2.9)$$

Performing the inverse canonical substitution we have

$$\psi(n, t) = \Omega(n, t)\mathbb{E}[\psi_0(N_n(t))/\Omega(N_n(t), 0)]$$
$$= \exp[(\lambda^2 - \tfrac{1}{2})\omega t]\mathbb{E}[\psi_0(N_n(t))\Omega(N_n(0), 0)/\Omega(N_n(t), 0)]. \qquad (2.10)$$

In this paper we will drop the contribution from the ground-state energy $\frac{1}{2}\omega$ of the free harmonic oscillator. Formula (2.10) will be the starting point of our approach to field-particle interactions.

3. Field–particle interaction

We now consider a non-relativistic particle in interaction with a field. As a first step we specialise the field to be a single oscillator. We write the Hamiltonian of this system in the following form:

$$H = \tfrac{1}{2}a^+a\omega + \tfrac{1}{2}p^2 + \lambda(q)\omega a^+ + \lambda^*(q)\omega a \qquad (3.1)$$

where a^+ and a now refer to the oscillator, q is the position operator of the non-relativistic particle, p is the conjugate momentum and $\lambda(q)$ is a complex function of q. Having defined $\psi(x, n, t) = \langle x, n|\psi_t\rangle$ we have the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(x, n, t) = n\omega\psi(x, n, t) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x, n, t)$$
$$+ \lambda^*(x)\omega(n+1)^{1/2}\psi(x, n+1, t) + \lambda(x)\omega\sqrt{n}\psi(x, n-1, t). \qquad (3.2)$$

The representation is mixed; in fact, $\psi(x, n, t)$ is the joint probability amplitude of finding n 'photons' and the particle in x. We want to give a probabilistic solution to this equation. First let us put $\lambda(x)$ in the usual form of field interactions:

$$\lambda(x) = \eta \exp(ikx) \tag{3.3}$$

with η being complex number independent of x. Now let us define

$$\Omega(x, n, t) \equiv \exp(i\omega|\eta|^2 t) \frac{(-\eta\, e^{ikx})^n}{(n!)^{1/2}}. \tag{3.4}$$

Then we perform the substitution $\psi(x, n, t) \to \phi(x, n, t) \equiv \psi(x, n, t)/\Omega(x, n, t)$.

The new function satisfies the equation (for imaginary time)

$$\frac{\partial}{\partial t}\phi(x, n, t) = -(n+|\eta|^2)\omega\phi(x, n, t) + n\omega\phi(x, n-1, t)$$
$$+\omega|\eta|^2\phi(x, n+1, t) + \tfrac{1}{2}\left(\frac{\partial}{\partial x} + ikn\right)^2 \phi(x, n, t). \tag{3.5}$$

In this form we have a random field $ikN_n(t)$ which appears explicitly as a correction of the Laplacian. The solution of this equation is given by

$$\phi(x, n, t) = \mathbb{E}\left[\phi_0(w_x(t), N_n(t)) \exp\left(ik \int_0^t N_n(r)\, dw_x\right)\right]. \tag{3.6}$$

The expectation is taken with respect to the Wiener process $w_x(t)$ starting in x at time $t=0$ and the usual birth and death process $N_n(t)$. We do not need to take particular care of the definition of the integral with respect to the Wiener process. We give a derivation of (3.6) in appendix 1.

Finally the solution of (3.2), for an imaginary time, is

$$\psi(x, n, t) = \exp(|\eta|^2\omega t)\mathbb{E}[\psi_0(w_x(t), N_n(t))$$
$$\times \exp\left(ik \int_0^t N_n(r)\, dw_x\right)\Omega(w_x(0), N_n(0), 0)/\Omega(w_x(t), N_n(t), 0)\Big]. \tag{3.7}$$

We are now able to tackle the problem of finding the probabilistic solution for the equation of a system of a non-relativistic particle in interaction with a field that is a set of harmonic oscillators. We have the general Hamiltonian

$$H = \tfrac{1}{2}|\boldsymbol{p}|^2 + \sum_k a_k^+ a_k \omega_k + \sum_k [\eta_k^* \omega_k \exp(-i\boldsymbol{k}\boldsymbol{x})a_k + \eta_k\omega_k \exp(i\boldsymbol{k}\boldsymbol{x})a_k^+] \tag{3.8}$$

where $\boldsymbol{x}$ is the vector position of the non-relativistic particle, $\boldsymbol{p}$ is the conjugate momentum and a_k^+, a_k are the annihilation and creation operators for the 'photon' of momentum $\boldsymbol{k}$ and energy ω_k. η_k and ω_k are unspecified functions of $k = |\boldsymbol{k}|$ (indeed, they may have a further dependence on other indices, but this does not produce any complications).

The probability amplitude to find the system in $\boldsymbol{x}$, $\hat{n}$ at time t is $\psi(\boldsymbol{x}, \hat{n}, t)$ where $\hat{n}$ denotes the set of positive integers n_k representing the photon number of mode $\boldsymbol{k}$.

By a generalisation of (3.7) we have (as usual for imaginary time)

$$\psi(\boldsymbol{x}, \hat{n}, t) = \exp\left(\sum_k |\eta_k|^2\omega_k t\right)\mathbb{E}\Big[\psi_0(\boldsymbol{w}_x(t), \hat{N}_{\hat{n}}(t))$$
$$\times \exp\left(i\sum_k \int_0^t N_k(r)\boldsymbol{k}\, d\boldsymbol{w}_x\right)\Omega(\boldsymbol{w}_x(0), \hat{N}_{\hat{n}}(0), 0)/\Omega(\boldsymbol{w}_x(t), \hat{N}_{\hat{n}}(t), 0)\Big]. \tag{3.9}$$

The expectation is taken with respect to $\hat{N}_{\hat{n}}(t)$ and $w_x(t)$ where $\hat{N}_{\hat{n}}(t)$ indicates the set of processes $N_k(t)$ starting in n_k at time $t=0$ and $w_x(t)$ is a three-dimensional Brownian motion. $\Omega(x, \hat{n}, t)$ is defined by

$$\Omega(x, \hat{n}, t)=\prod_k \exp(|\eta_k|^2\omega_k t)\frac{(-\eta_k \exp(\mathrm{i}kx))^{n_k}}{(n_k!)^{1/2}}. \tag{3.10}$$

To interpret (3.9) we have to remember that the intensities for each process $N_k(t)$ are $\omega_k|\eta_k|^2$ for emission and $n_k\omega_k$ for absorption respectively.

The integral with respect to the Wiener variable can be integrated by parts

$$\int_0^t N_k(r)\mathrm{d}w_x=(N_k(t)w_x(t)-N_k(0)w_x(0))-\int_0^t w_x(r)\,\mathrm{d}N_k \tag{3.11}$$

and we obtain

$$\psi(x, n, t)=\exp\left(\sum_k|\eta_k|^2\omega_k t\right)\mathbb{E}\bigg[\psi_0(w_x(t), \hat{N}_{\hat{n}}(t))$$
$$\times\exp\left(-\mathrm{i}\sum_k\int_0^t k w_x(r)\,\mathrm{d}N_k\right)\Omega(0, \hat{N}_{\hat{n}}(0), 0)/\Omega(0, \hat{N}_{\hat{n}}(t), 0)\bigg]. \tag{3.12}$$

We can derive a first inequality from (3.12)

$$|\psi(x, \hat{n}, t)|\leq\mathbb{E}[|\psi_0(w_x(t), \hat{N}_{\hat{n}}(t))|\,|\Omega(0, \hat{N}_{\hat{n}}(0), 0)|$$
$$\times|\Omega(0, \hat{N}_{\hat{n}}(t), 0)|^{-1}]\exp\left(\sum_k|\eta_k|^2\omega_k t\right). \tag{3.13}$$

From this inequality we deduce that the lowest eigenvalue E_0 of the Hamiltonian (3.8) is larger than the lowest eigenvalue E_0' of the Hamiltonian H' which is responsible for the evolution of the right-hand side of (3.13). It is easy to check that

$$H'=\tfrac{1}{2}|p|^2+\sum_k a_k^+a_k\omega_k-\sum_k\omega_k|\eta_k|(a_k^++a_k). \tag{3.14}$$

In fact, now, particle and oscillators are uncoupled so that the right-hand side of (3.13) is the solution of the Schrödinger equation associated to H' when the initial condition is $|\psi_0(x, \hat{n})|$.

The minimum energy of the free particle is zero while the minimum energy of the oscillators can be written utilising the results of § 2. We have finally

$$E_0\geq E_0'=-\sum_k|\eta_k|^2\omega_k. \tag{3.15}$$

We point out that the Hamiltonian (3.8) is very general and we are able to cover many different physical interactions by assuming different k dependence of the functions η_k and ω_k.

4. Elimination of the particle's variables

One of the main successes of the probabilistic formulation of quantum mechanics lies in the possibility of partial elimination of variables of the system. For example, we could specialise the expressions (3.9) and (3.12) to the case for which in the initial and final state there are no 'photons' and then integrate with respect to all the measures of birth and death processes.

In this case we are left with an expectation with respect to the Wiener process only. This possibility has already been explored with the usual techniques of path integration. We are now interested in the inverse procedure: we will eliminate the particle variables and then we try to extract information from the resulting expectation with respect to the 'photons' variables.

We start by rewriting equation (3.9) with the assumption that the particle has a position $\boldsymbol{x}'$ at the initial time. We have

$$\psi_0(\boldsymbol{x}, \hat{n}) \equiv \psi_0(\hat{n})\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{4.1}$$

and so

$$\psi(\boldsymbol{x}, \hat{n}, t) = \exp\left(\sum_k |\eta_k|^2 \omega_k t\right)\mathbb{E}\left[\psi_0(\hat{N}_{\hat{n}}(t))\delta(\boldsymbol{w}_x(t)-\boldsymbol{x}')\right.$$
$$\left.\times \exp\left(\mathrm{i}\sum_k \int_0^t N_k(r)\boldsymbol{k}\,\mathrm{d}\boldsymbol{w}_x\right)\Omega(\boldsymbol{w}_x(0), \hat{N}_{\hat{n}}(0), 0)/\Omega(\boldsymbol{w}_x(t), \hat{N}_{\hat{n}}(t), 0)\right]. \tag{4.2}$$

We can extract from this the expectation with respect to the Wiener process

$$\mathbb{E}\left[\delta(w_x(t)-x')\exp\left(\mathrm{i}\sum_k \int_0^t N_k(r)k\,\mathrm{d}w_x\right)\right]. \tag{4.3}$$

Here the expression $\mathrm{i}\,\Sigma_k\, N_k(r)\boldsymbol{k}$ appears as a time-dependent unspecified function so that we are able to calculate explicitly (4.3)

$$\frac{1}{(2\pi t)^{1/2}}\exp\left[-\frac{1}{2t}\left(x-x'+\mathrm{i}\sum_k\int_0^t N_k(r)k\,\mathrm{d}r\right)^2 - \tfrac{1}{2}\int_0^t\left(\sum_k N_k(r)k\right)^2\mathrm{d}r\right]. \tag{4.4}$$

We can furthermore specialise (4.4) choosing $x = x'$ the above expression turns into

$$\frac{1}{(2\pi t)^{1/2}}\exp(S_i(t)) = \frac{1}{(2\pi t)^{1/2}}\exp\left\{\frac{1}{2}\left[\frac{1}{t}\left(\sum_k\int_0^t N_k(r)k\,\mathrm{d}r\right)^2 - \int_0^t\left(\sum_k N_k(r)k\right)^2\mathrm{d}r\right]\right\} \tag{4.5}$$

which is a real positive expression, and it can be thought of as generated by a potential non-local in time that describes the interaction between 'photons'. We can now write the transition probability amplitude that the system goes from the state of zero 'photons' to the state of zero 'photons'. Using (4.2) and (4.5) we obtain

$$P(t) = \frac{1}{(2\pi t)^{1/2}}\mathbb{E}[\exp(S_i(t))\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]\exp\left(\sum_k |\eta_k|^2\omega_k t\right) \tag{4.6}$$

the expectation being now taken with respect to the processes $N_k(t)$ starting in 0 at time $t = 0$.

We are interested in this expression because we are able to extract some information about the ground-state energy of the Hamiltonian (3.8). In fact, it follows from the general theory of quantum mechanics that (for imaginary time)

$$E_0 = \lim_{t\to+\infty} -\frac{1}{t}\log P(t). \tag{4.7}$$

Here we will drop the unimportant factor $(2\pi t)^{-1/2}$ in $P(t)$.

We can extract some preliminary information from (4.6) and (4.7) very easily. Let us remark that every possible 'trajectory' of the process $\hat{N}_{\hat{0}}(t)$ gives a positive contribution to $P(t)$, so that if we restrict ourselves to a certain set of 'trajectories' we obtain a value which is lower than $P(t)$. In particular, let us consider the contribution of the 'trajectory' in which, for every $\boldsymbol{k}$, $N_{\boldsymbol{k}}(r)=0$ in the interval $0\leqslant r\leqslant t$. We can easily see that the probability of this event is

$$\prod_{\boldsymbol{k}} \exp(-|\eta_k|^2\omega_k t). \tag{4.8}$$

This result is understandable if we remember that the intensity of a jump from the ground state is $|\eta_k|^2\omega_k$ for a single oscillator of momentum $\boldsymbol{k}$. Taking into account that for the 'trajectory' that we consider, $S_i(t)=0$, we are now able to write

$$P(t)\geqslant \exp\left(\sum_{\boldsymbol{k}} |\eta_k|^2\omega_k t\right)\prod_{\boldsymbol{k}} \exp(|\eta_k|^2\omega_k t)=1 \tag{4.9}$$

and from (4.7) we deduce that $E_0\leqslant 0$. Indeed, this is still not a good result and it is possible to do much better using the property of convexity of the exponential function.

Let us define

$$\mathscr{E}=\sum_{\boldsymbol{k}} |\eta_k|^2\omega_k. \tag{4.10}$$

We then rewrite (4.6) in the form

$$P(t)=\mathbb{E}[\exp(S_i(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]. \tag{4.11}$$

We can obtain a first inequality taking into account that the action $S_i(t)$ defined in (4.5) is the sum of a positive part $S^1(t)$ and a negative part $S^0(t)$:

$$\begin{aligned} S^1(t)&\equiv\frac{1}{2t}\left(\sum_{\boldsymbol{k}}\int_0^t N_{\boldsymbol{k}}(r)\boldsymbol{k}\,\mathrm{d}r\right)^2\\ S^0(t)&\equiv-\frac{1}{2}\int_0^t\left(\sum_{\boldsymbol{k}} N_{\boldsymbol{k}}(r)\boldsymbol{k}\right)^2\mathrm{d}r. \end{aligned} \tag{4.12}$$

From equation (4.11) follows

$$P(t)\geqslant\mathbb{E}[\exp(\delta_0(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]\equiv P'(t). \tag{4.13}$$

This simplification of the action is not really necessary but it does not worsen the bound of the energy we will calculate.

Now let us assume that the sums extend over $\boldsymbol{k}$ as well as $-\boldsymbol{k}$ so that for every oscillator of momentum $\boldsymbol{k}$ we have the oscillator of momentum $-\boldsymbol{k}$. We can write

$$\begin{aligned} S^0(t)&\equiv S(t)+S'(t)\equiv\sum_{\boldsymbol{k}} S_{\boldsymbol{k}}(t)+S'(t)\\ S'(t)&\equiv\sum_{\boldsymbol{k}\neq\pm\boldsymbol{k}'}\sum_{\boldsymbol{k}'}-\frac{1}{2}\int_0^t N_{\boldsymbol{k}}(r)N_{\boldsymbol{k}'}(r)\boldsymbol{k}\boldsymbol{k}'\,\mathrm{d}r\\ S_{\boldsymbol{k}}(t)&\equiv-\frac{1}{4}\int_0^t (N_{\boldsymbol{k}}(r)-N_{-\boldsymbol{k}}(r))^2k^2\,\mathrm{d}r \end{aligned} \tag{4.14}$$

and we have

$$\frac{\mathbb{E}[S'(t)\exp(S(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]}{\mathbb{E}[\exp(S(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]}=\langle S'(t)\rangle=0. \tag{4.15}$$

In fact, the intensities of the processes $N_k(t)$ and $N_{-k}(t)$ are the same and the action $S(t)+\mathscr{E}t$ is symmetric with respect to the inversion of momentum. This implies for $\boldsymbol{k}\neq\pm\boldsymbol{k}'$:

$$\mathbb{E}[N_k(r)(N_{k'}(r)-N_{-k'}(r))\boldsymbol{k}\cdot\boldsymbol{k}'\exp(S(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]=0 \tag{4.16}$$

and from this it is easy to obtain (4.15). It is clear now why we had to spread the action $S^0(t)$ into $S(t)$ and $S'(t)$. In fact, the equality (4.16) does not hold for $\boldsymbol{k}=\pm\boldsymbol{k}'$. Now using the properties of convexity of the exponential function we are able to write

$$P'(t)\geqslant\mathbb{E}[\exp(S(t)+\mathscr{E}t+\langle S'(t)\rangle)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]=\mathbb{E}[\exp(S(t)+\mathscr{E}t)\delta_{\hat{0},\hat{N}_{\hat{0}}(t)}]. \tag{4.17}$$

We have, at this point, reduced the study of a field with many complicated interactions to the study of a set of uncoupled pairs of oscillators, each pair being composed by an oscillator with momentum $\boldsymbol{k}$ and an oscillator with momentum $-\boldsymbol{k}$ in interaction.

In other words we reduce the problem to the study of

$$P_k(t)\equiv\mathbb{E}_{\pm k}[\exp(2S_k(t))\delta_{0,N_k(t)}\delta_{0,N_{-k}(t)}] \tag{4.18}$$

where the expectation is taken with respect to the processes $N_k(t)$ and $N_{-k}(t)$ and $S_k(t)$ is given by the third relation in (4.14). We can now write

$$P(t)\geqslant P'(t)\geqslant\left(\prod_k P_k(t)\right)^{1/2}\exp(\mathscr{E}t). \tag{4.19}$$

We are not able to calculate the expression (4.18) exactly but we can give some inequalities introducing the trial action

$$S_k^A(t)=-A(k)\int_0^t N_k(r)\,\mathrm{d}r \tag{4.20}$$

where $A(k)$ is an arbitrary function of $k=|\boldsymbol{k}|$ which will be chosen in order to obtain the best estimates. We have

$$[P_k(t)]^{1/2}\geqslant\mathbb{E}_k[\exp(\langle S_k(t)-S_k^A(t)\rangle+S_k^A(t))\delta_{0,N_k(t)}]. \tag{4.21}$$

The expectation is now taken with respect to the process $N_k(t)$ alone, and

$$\langle S_k(t)-S_k^A\rangle=-\frac{1}{2}\int_0^t(\langle N_k^2(r)\rangle-\langle N_k(t)\rangle^2)k^2\,\mathrm{d}r+A(k)\int_0^t\langle N_k(r)\rangle\,\mathrm{d}r \tag{4.22}$$

where $\langle N_k^2(r)\rangle$ and $\langle N_k(r)\rangle^2$ are given as mean values over all the 'trajectories' distorted by the trial action and with the initial and final states being in the zero-photon configurations.

The right-hand side of expression (4.21) is calculated in appendix 2 and we obtain from (4.7), (4.19) and (4.21)

$$E_0\leqslant\sum_k|\eta_k|^2\omega_k^2(\tfrac{1}{2}k^2-\omega_k-2A(k))(\omega_k+A(k))^{-2}. \tag{4.23}$$

The function $A(k)$ can now be chosen in order to minimise the right-hand side of (4.23). Such a choice should be $A(k)=\frac{1}{2}k^2$. Rewriting (4.23) together with inequality (3.15) we finally have

$$-\sum_k|\eta_k|^2\omega_k\leqslant E_0\leqslant-\sum_k|\eta_k|^2\omega_k^2/(\omega_k+\tfrac{1}{2}k^2) \tag{4.24}$$

which is a general result. As an example of the application of (4.24) we can consider the case of the polaron; we have $|\eta_k|^2 = 2\sqrt{2}\alpha\pi/k^2$, $\omega_k = 1$ (the extra factor two is a consequence of the spin of the electron). With the replacement $\Sigma_k \to \int d^3k/(2\pi)^3$ inequality (4.24) gives $E_0 \leq \alpha$ which is a well known estimate of the polaron theory.

5. Conclusions

The main practical result of this paper is the statement of inequality (4.24) where E_0 is the ground-state energy of a system described by the general Hamiltonian (3.8).

After the work was finished we learnt that the bound (4.24) was already found (see, for example, Spohn 1986) utilising standard functional methods. However we think that our approach is of interest because it gives a new description of quantum fields that can be considered as complementary to the usual one. This also means that our theory provides new instruments to investigate polaron theories. We think, for example, of the possibility of obtaining better estimates of the ground-state energies. This seems feasible: in fact we observe that as an intermediate step in the calculation of (4.24) we give a bound for the expression (4.18), which can probably be calculated exactly. Work in this direction is in progress.

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Appendix 1. A derivation of equation (3.6)

Let us consider the equation

$$\frac{\partial}{\partial t}\phi(x,n,t) = -(|\lambda|^2+n)\omega\phi(x,n,t) + n\omega\phi(x,n-1,t) + \omega|\lambda|^2\phi(x,n+1,t) + \tfrac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}kn\right)^2\phi(x,n,t). \tag{A1.1}$$

We want to show that

$$\phi(x,n,t) = \mathbb{E}\left[\phi_0(w_x(t), N_n(t))\exp\left(\mathrm{i}k\int_0^t N_n(r)\,\mathrm{d}w_x(r)\right)\right] \tag{A1.2}$$

where the expectation is taken with respect to the Wiener process $w(t)$ and the birth and death process $N_n(t)$. The expression $\phi(x, n, t)$ satisfies the initial condition

$$\phi(x,n,0) = \phi_0(x,n). \tag{A1.3}$$

We start taking the expectation (A1.2) at time $t+\Delta t$

$$\phi(x,n,t+\Delta t) = \mathbb{E}\left[\phi_0(w_x(t+\Delta t), N_n(t+\Delta t))\exp\left(\mathrm{i}k\int_0^{t+\Delta t} N_n(r)\,\mathrm{d}w_x(r)\right)\right]. \tag{A1.4}$$

Let us state the following equalities:

$$\begin{aligned} w_x(t+\Delta t) &= w_0(t+\Delta t)+x = w_0(t+\Delta t)-w_0(\Delta t)+w_0(\Delta t)+x \\ &= w_0'(t)+w_0(\Delta t)+x = w_x'(t)+w_0(\Delta t) \\ N_n(t+\Delta t) &= N'(t, N_n(\Delta t)) \end{aligned} \tag{A1.5}$$

where $w_0'(t)$ and $w_0(\Delta t)$ are two independent Wiener variables starting in 0 at time $t=0$ and $N'(t, N_n(\Delta t))$ is a birth and death process with the same transition probabilities as $N_n(t)$ but starting in $N_n(\Delta t)$ at time $t=0$. Taking into account that $\mathrm{d}w_x = \mathrm{d}w_0$ we can write

$$\begin{aligned} \int_0^{t+\Delta t} N_n(r)\,\mathrm{d}w_x(r) &= \int_{\Delta t}^{t+\Delta t} N_n(r)\,\mathrm{d}w_x(r) + \int_0^{\Delta t} N_n(r)\,\mathrm{d}w_x(r) \\ &= \int_0^{t} N_n(r+\Delta t)\,\mathrm{d}w_0(r+\Delta t) + \int_0^{\Delta t} N_n(r)\,\mathrm{d}w_0(r) \\ &= \int_0^{t} N'(r, N_n(\Delta t))\,\mathrm{d}w_0'(r) + \int_0^{\Delta t} N_n(r)\,\mathrm{d}w_0(r). \end{aligned} \tag{A1.6}$$

The second integral of the last line is equal to $nw_0(\Delta t)$ up to terms of order $o(\Delta t)$.

Expanding (A1.4) with respect to the 'small' $w_0(\Delta t)$ and taking the expectation with respect to this variable we obtain

$$\begin{aligned} \phi(x, n, t+\Delta t)&\left[1+\tfrac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}kn\right)^2\Delta t\right]\mathbb{E}\bigg[\phi_0(w_x'(t), N'(t, N_n(\Delta t))) \\ &\times \exp\left(\mathrm{i}k\int_0^t N'(r, N_n(\Delta t))\,\mathrm{d}w_0'(r)\right)\bigg] + o(\Delta t). \end{aligned} \tag{A1.7}$$

Taking into account that

$$\phi(x, n, t) = \mathbb{E}\left[\phi_0(w_x'(t), N'(t, n))\exp\left(\mathrm{i}k\int_0^t N'(r, n)\,\mathrm{d}w_0'(r)\right)\right] \tag{A1.8}$$

and taking the expectation with respect to $N_n(\Delta t)$ we finally have

$$\begin{aligned} \phi(x, n, t+\Delta t) &= \left[1+\tfrac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}km\right)^2\Delta t - (|\lambda|^2+n)\omega\Delta t\right]\phi(x, n, t) \\ &\quad + |\lambda|^2\omega\phi(x, n+1, t)\Delta t + n\omega\phi(x, n-1, t)\Delta t + o(\Delta t). \end{aligned} \tag{A1.9}$$

In the limit $\Delta t \to 0$ (A1.1) follows.

Appendix 2. Calculation of expectation (4.21)

In order to calculate the right-hand side of (4.21) we will solve the equation

$$\frac{\partial}{\partial t}\psi(n, t) = -\omega(n+|\eta|^2)\psi(n, t) + \omega|\eta|^2\psi(n+1, t) + n\omega\psi(n-1, t) - An\psi(n, t) \tag{A2.1}$$

satisfied by

$$\psi(n, t) \equiv \mathbb{E}\left[\exp\left(-A\int_0^t N_n(r)\,\mathrm{d}r\right)\psi_0(N_n(t))\right]. \tag{A2.2}$$

First we perform the transformation

$$\psi(n,t)\to\phi(n,t)\equiv\exp\left[\left(\omega-\frac{\omega^2}{\omega+A}\right)|\eta|^2 t\right]\left(\frac{A+\omega}{\omega}\right)^n\psi(n,t). \tag{A2.3}$$

For the new function we have the equation

$$\frac{\partial}{\partial t}\phi(n,t)=-\left((\omega+A)n+\frac{|\eta|^2\omega^2}{\omega+A}\right)\phi(n,t)+\frac{|\eta|^2\omega^2}{\omega+A}\phi(n+1,t)+n(\omega+A)\phi(n-l,t). \tag{A2.4}$$

This is now a backward Kolmogorov equation for a birth and death process with intensities $(\omega+A)n$ and $|\eta|^2\omega^2/(\omega+A)$ for absorption and emission respectively. Note that

$$\phi(n,t)=\sum_{m=0}^{\infty}p'(m,t;n,0)\phi_0(m) \tag{A2.5}$$

where the transition probability $p'(m, t; n, 0)$ is given by (2.7) once we replace ω by $\omega+A$ and λ^2 by $|\eta|^2\omega/(\omega+A)$.

Because of (A2.3) we have

$$\psi(n,t)=\left(\frac{\omega}{\omega+A}\right)^n\exp\left[\left(\frac{\omega^2}{\omega+A}-\omega\right)|\eta|^2 t\right]\sum_{m=0}^{\infty}p'(m,t;n,0)\left(\frac{\omega+A}{\omega}\right)^m\psi_0(m) \tag{A2.6}$$

so the transition probability amplitude connected with equation (A2.1) is

$$p''(m,t;n,0)=\left(\frac{\omega+A}{\omega}\right)^{m-n}\exp\left[\left(\frac{\omega^2}{\omega+A}-\omega\right)|\eta|^2 t\right]p'(m,t;n,0). \tag{A2.7}$$

We have now the ingredients to calculate the right-hand side of (4.21). We define $p''_k(m, t; n, 0)$ taking the expression (A2.7) and substituting ω, A, η with ω_k, $A(k)$, η_k. It turns out that

$$\begin{aligned}\langle N_k(r)\rangle&=\sum_{m=0}^{\infty}p''_k(0,t;m,r)mp''_k(m,r;0,0)/p''_k(0;t;0,0)\\&=\frac{|\eta_k|^2\omega_k^2}{[\omega_k+A(k)]^2}\{1-\exp[-(\omega_k+A(k))(t-r)]\}\{1-\exp[-(\omega_k+A(k))r]\}.\end{aligned} \tag{A2.8}$$

In the same way we obtain

$$\langle N_k^2(r)\rangle=\langle N_k(r)\rangle^2+\langle N_k(r)\rangle. \tag{A2.9}$$

We are now able to calculate the expectation (4.21). In fact, taking into account that

$$\langle S_k(t)-S_k^{A}(t)\rangle=(A(k)-\tfrac{1}{2}k^2)\int_0^t\langle N_k(r)\rangle\,\mathrm{d}r \tag{A2.10}$$

we can write

$$\begin{aligned}&\mathbb{E}_k[\exp(S_k^{A}(t))\delta_{0,N_k(t)}]\exp(\langle S_k(t)-S_k^{A}(t)\rangle)\\&\quad=p''_k(0,t;0,0)\exp\left((A(k)-\tfrac{1}{2}k^2)\int_0^t\langle N_k(r)\rangle\,\mathrm{d}r\right).\end{aligned} \tag{A2.11}$$

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